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A line sink in a rotating stratified fluid

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The two-dimensional flow of an unbounded rotating stratified fluid towards a link sink is studied. The initial-value problem of suddenly initiating the sink flow is solved in Laplace space for a non-diffusive, inviscid fluid using the linearized Boussinesq equations. The solution shows that the sink flow is established by inertio–gravity waves radiated from the sink and that the initial development of the flow depends critically on the ratio of the inertial frequency, f, to the buoyancy frequency, N. For $f < N$ the flow collapses to a horizontal withdrawal layer structure. The final steady state resembles potential flow in which the vertical axis is shrunk by a factor of f/N with a superimposed azimuthal velocity. Viscous, diffusive and nonlinear effects are studied using scaling analysis. A classification scheme based on two parameters delineating various force balance regimes and giving the corresponding withdrawal layer thicknesses is presented. The results show that under certain conditions rotation may cause a thicker withdrawal layer than would be observed if there were no rotation.

1. Introduction

This study aims to shed light on the nature of the flow field caused by a sink in a rotating stratified fluid. Specifically, the analysis examines the time development and final steady state of the sink flow. It is motivated by the proposition that some sink flows in geophysical and limnological situations may be influenced by rotation as well as density stratification (Imberger 1980; Whitehead 1980). One example is the withdrawal of water from reservoirs which are stratified and are large enough so that rotation may influence the withdrawal dynamics. Indeed, it has been suggested (Imberger 1980) that rotational effects may be responsible for the discrepancies between field observations of withdrawal layer thicknesses and those predicted by non-rotating theory. Another example is the entrainment of ambient fluid by a plume where the plume may be considered to act as a vertically distributed sink. In deep ocean convection events which involve the sinking of large masses of dense water, usually at high latitudes, entrainment of surrounding fluid may be on a scale large enough for rotation to be important (for a review of this subject see Killworth 1983).

It is known that the flow of a stratified non-rotating fluid towards a sink, due to the action of buoyancy forces, comes from a narrow layer at the level of the sink. This is called selective withdrawal and, given its importance to water quality in the reservoir withdrawal problem, is a much studied phenomenon from theoretical, laboratory and field studies. For a review of this subject see Imberger & Patterson (1990). By analysing the initial-value problem of suddenly initiating the sink flow of a linearly stratified fluid contained in a horizontal duct, Pao & Kao (1974) showed that the withdrawal layer is established by waves of zero frequency but finite group velocity travelling outward from the sink. These waves are called columnar disturbances or shear waves and act to modify the initial potential flow structure, leaving a withdrawal layer in their wake of thickness of order of the vertical wavelength of the shear wave. Given the analogy between the dynamics of rotating homogeneous fluids and stratified non-rotating fluids (Veronis 1967) similar waves are responsible for the vertical withdrawal layer above a sink in a rotating homogeneous fluid. Such vertical withdrawal layers in rotating homogeneous fluids have been studied experimentally, e.g. Shih & Pao (1971). Bretherton (1967) showed that such shear waves are responsible for Taylor column formation in rotating fluids and, analogously, blocking in stratified fluids.

The combined problem of sink flow in a fluid that is both rotating and stratified has received little attention in the literature despite its obvious oceanographic and limnological importance. Axisymmetric sink flow in a rotating stratified fluid has been studied by Whitehead (1980) using a quasi-steady analysis and a two-layer fluid approximation to the stratified withdrawal problem in which the fluid in the withdrawal layer is of different density to the surrounding fluid. He does not study the time evolution of the flow from an initial quiescent state but rather considers the modification of an initial radial inflow, due solely to stratification, as rotation is turned on. Whitehead did not find a steady state but rather a withdrawal layer which grew in thickness like $t^{\frac{1}{3}}$. Such a growth in layer thickness with time is contrary to the findings of this paper which considers the initiation of the sink flow from rest.

Monismith & Maxworthy (1989) performed experiments on withdrawal from a rotating stratified fluid concentrating on the regime where stratification dominates rotation. The fluid was withdrawn via a point sink located at the end of a rectangular tank. A withdrawal layer formed in which fluid propagated anticyclonically around the perimeter of the tank. They explained their results in terms of 'Kelvin shear wave' dynamics which are essentially shear waves modified by the effects of rotation. Such a theory requires the presence of the lateral boundaries in order for these Kelvin shear waves to exist (and hence is not applicable to the open ocean problem). They also found there was no withdrawal layer thickening attributable to rotation. It is not clear, however, whether this is a general statement about the effect of rotation on the withdrawal problem or a consequence of the parameter regime in which their experiments were performed.

Here a much simplified approach to the problem is taken in order to understand some of the dynamics involved in the flow of a rotating stratified fluid towards a sink and consequently the possible effect of rotation on the withdrawal problem. The word possible is used here since the effect of side boundaries is not included in the following analysis. The presence of such boundaries has a significant effect on rotating stratified flows (e.g. Huppert & Stern 1974) and a full analysis of the initial-value problem incorporating them is complicated. However, it is likely that, based on Gill's (1976) study of gravitational adjustment in a rotating channel, sidewall effects are limited to within a Rossby radius of deformation from the wall and that further away from the sidewall the withdrawal flow evolves as if no such boundaries are present. Since the eventual steady state, if any, is unknown the problem is formulated as an initial-value problem in which the sink flow is started up from rest. This is done in §2 for an unbounded, rotating, stratified fluid using linearized inviscid dynamics. In the spirit of Bretherton (1967) it is formulated in two dimensions for ease of analysis and although this configuration is difficult to achieve in the laboratory, the physics of the flow should be essentially the same as would be observed for the axisymmetric case. The sink is represented as a forcing term in the conservation of mass equation and is of infinitesimal width, i.e. is represented by a product of Dirac delta functions. The implication is that the solution is interpreted as a Green's function or in a distributional sense. As will be shown, the solution contains spatially varying persistent oscillations as $t \to \infty$ which are then interpreted as distributions and enables statements such as $\sin \omega t/\omega \to \pi\delta(\omega)$ as $t \to \infty$ to be made. This is highlighted by Hendershott (1969) who studied impulsively started oscillations by a spherical source in a rotating stratified fluid. In his solution the velocity field at large times included a component which decayed like $t^{-\frac{1}{2}}$ and was attributable to the interaction of inertio-gravity waves arriving at a particular location from both sides of the finite-sized source. Such an effect will not be observed here since the sink is of infinitesimal width but, in principle, one could use an appropriate distribution of sources to generate the two-dimensional analogue of Hendershott's problem. This would involve a spatial integral, the persistent oscillations of which would decay according to some fractional power of t which could be determined by, for example, the method of stationary phase.

Section 3 examines the case where the sink flow is impulsive and the velocity field, caused by the radiation of inertio-gravity waves, is derived in detail. The solution is then used to explain the development of the steady-state flow for a sink of constant strength. Section 4 includes the effects of nonlinearity, viscosity and species diffusion in determining the final steady state using scaling analysis in which the classification scheme of Imberger, Thompson & Fandry (1976) is generalized to include the effect of rotation. The conclusions are presented in §5.

2. The initial-value problem

2.1. *Problem formulation and solution*

The equations governing the motions of an incompressible stratified fluid on an f-plane form the starting point of this study. They are (see Walin 1969)

$$\rho_T(\mathrm{D}\boldsymbol{v}/\mathrm{D}t + f\boldsymbol{k}\times\boldsymbol{v}) = -\boldsymbol{\nabla} p_T - \rho_T g\boldsymbol{k} + \mu\nabla^2\boldsymbol{v}, \tag{2.1}$$

$$\mathrm{D}\rho_T/\mathrm{D}t = \kappa\nabla^2\rho_T, \tag{2.2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{2.3}$$

where $$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad \text{and} \quad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Here $\boldsymbol{v} = (u, v, w)$ is the velocity vector in the (x, y, z) directions, p_T is the pressure, ρ_T is the density, μ is the dynamic viscosity, κ the diffusivity of the stratifying species, g the acceleration due to gravity and $\boldsymbol{k}$ is a unit vector in the z-direction.

As a particular case consider an inviscid, non-diffusive fluid occupying all space which is stably stratified with constant buoyancy frequency N. A line sink lies at the origin of the (x, z)-plane aligned along the y-axis and so the properties of the fluid motion are assumed to be independent of the transverse (y) coordinate and therefore may be considered two-dimensional. See figure 1 for the geometry of the flow domain.

Initially the fluid is at rest in this coordinate system. At time $t = 0$ a sink with strength $q(t)$ (volume flux per unit length in the y direction), $t \geqslant 0$, is suddenly switched on.

In the case $f < N$ internal wave activity will first be evident on a timescale N^{-1} after the sink has been turned on. This gives a natural timescale of N^{-1} and natural

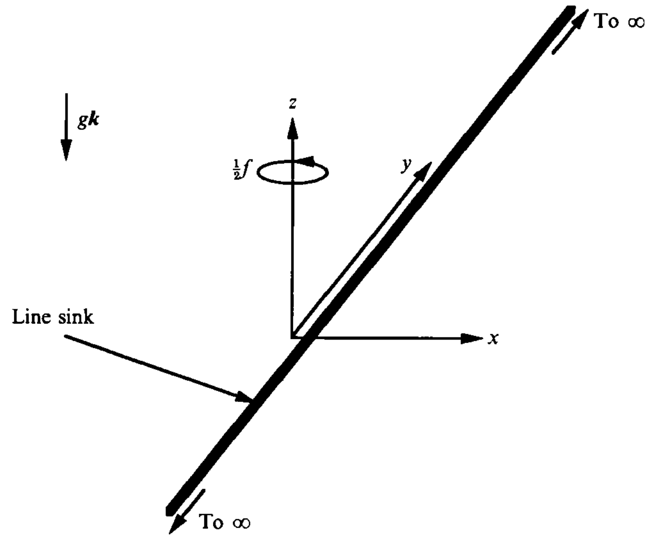

FIGURE 1. Coordinate system showing the orientation of the line sink.

lengthscale $(q/N)^{\frac{1}{2}}$. The nonlinear advective terms may be ignored relative to unsteady inertial terms by requiring that distances from the sink L are such that $L \gg (q/N)^{\frac{1}{2}}$, or, equivalently, by requiring that the Froude number, $F = q/NL^2$, be small. A similar argument applies if $f > N$ by requiring the Rossby number to be small. Assuming that distances are sufficiently far from the sink, i.e. $F \ll 1$, then (2.1)–(2.3) may be linearized, together with the above assumptions of inviscidness and non-diffusiveness to give

$$u_x + w_z = -q(t)\,\delta(x)\,\delta(z), \tag{2.4}$$

$$\rho_0(u_t - fv) = -P_x, \tag{2.5}$$

$$v_t + fu = 0, \tag{2.6}$$

$$\rho_0 w_t = -P_z - g\rho, \tag{2.7}$$

$$\rho_t + w\rho_{0z} = 0. \tag{2.8}$$

Here u is the velocity component in the x-direction, w is the velocity component in the z-direction, v the azimuthal velocity parallel to the axis of the sink, P is the pressure perturbation from hydrostatic pressure and $\rho(x,z,t)$ is the variation of the density from the undisturbed density $\rho_0(z)$ i.e. $\rho_T = \rho_0 + \rho(x,z,t)$. The density perturbation is assumed small compared to ρ_0. The sink at the origin is represented by the delta function product in (2.4). The buoyancy frequency N is defined by $N^2 = -(g/\rho_0)(\mathrm{d}\rho_0/\mathrm{d}z)$ and is assumed to be constant. Just after the sink flow has been turned on rotation and stratification have no effect since their influence is manifested through forces proportional to the displacement of fluid particles (Veronis 1967) which is zero initially. Hence, since the fluid is incompressible, the appropriate initial condition to use is that of potential flow and, in particular, the potential flow caused by a link sink. Thus at $t = 0$, $u = \psi_z$, $w = -\psi_x$, $v = 0$ and $\rho = 0$, where $\psi = -(q(0)/2\pi$ arctan (z/x). Alternatively, in terms of the velocity potential, the initial conditions are $u = \phi_x$, $w = \phi_z$ where $\phi = -(q(0)/2\pi \log[(x^2+z^2)^{\frac{1}{2}}]$. The velocity components are required to vanish at infinity, i.e. $u, w \to 0$ as $x^2 + z^2 \to \infty$.

Following Hendershott (1969), the Laplace transform in the time variable of (2.4)–(2.8) is taken, where the Laplace transform has the usual definition:

$$\bar{g}(s) = \int_0^\infty g(t)\, e^{-st}\, dt.$$

The quantity $\bar{p} = \bar{P} - \rho_0 \phi$ is introduced in order to satisfy the initial conditions. Thus $\bar{p}(x, z, s)$ represents the difference between the transformed pressure perturbation $\bar{P}$ and the initial pressure written in terms of the velocity potential. Eliminating $\bar{v}$ and $\bar{\rho}$ from the transformed version of (2.4)–(2.8) gives

$$\rho_0(s^2+f^2)\bar{u} = -s\bar{p}_x, \qquad (2.9)$$

$$\rho_0(s^2+N^2)\bar{w} = -s\bar{p}_z. \qquad (2.10)$$

The boundary conditions at infinity are then $\bar{p}_x, \bar{p}_z \rightarrow 0$ as $x^2+z^2 \rightarrow \infty$. Substitution of (2.9) and (2.10) into the transformed version of (2.4) gives

$$\frac{s}{s^2+f^2}\bar{p}_{xx} + \frac{s}{s^2+N^2}\bar{p}_{zz} + \bar{w}\frac{d\rho_0}{dz} = \rho_0\, \bar{q}(s)\, \delta(x)\, \delta(z).$$

Since the buoyancy frequency, as defined earlier, is assumed to be constant it follows that the density increases exponentially with depth and can be written in the form $\rho_0 = A \exp(-\beta z)$ where A is a reference density (density at the level of the sink) and $\beta = N^2/g$. Using this form for the density and multiplying the above equation by $(s^2+f^2)/s$ the following equation for $\bar{p}$ is obtained:

$$\bar{p}_{xx} + \frac{s^2+f^2}{s^2+N^2}\bar{p}_{zz} + \beta\frac{s^2+f^2}{s^2+N^2}\bar{p}_z = \rho_0\, \bar{q}(s)\frac{s^2+f^2}{s}\delta(x)\, \delta(z). \qquad (2.11)$$

The Boussinesq approximation is now made by requiring that vertical distances H from the sink are such that $N^2H/g \ll 1$ (Hendershott 1969). If the vertical coordinate is scaled by H then the ratio of the third to second term on the right-hand side of (2.11) is $\beta H = N^2H/g$, which is assumed to be small in the Boussinesq limit. For a typical reservoir operation $\beta H = O(10^{-3})$.

Given the above restrictions (2.11) reduces to, in the Boussinesq limit,

$$\bar{p}_{xx} + \frac{s^2+f^2}{s^2+N^2}\bar{p}_{zz} = \rho_0\, \bar{q}(s)\frac{s^2+f^2}{s}\delta(x)\, \delta(z), \qquad (2.12)$$

where now ρ_0 is the density at the level of the sink, i.e. $\rho_0 = A$. Equation (2.12) can be rescaled into Poisson's equation by letting

$$z' = \left(\frac{s^2+N^2}{s^2+f^2}\right)^{\frac{1}{2}} z,$$

for which (2.12) becomes

$$\bar{p}_{xx} + \bar{p}_{z'z'} = \rho_0\, \bar{q}(s)\frac{s^2+f^2}{s}\delta(x)\, \delta\left[\left(\frac{s^2+f^2}{s^2+N^2}\right)^{\frac{1}{2}} z'\right]$$

$$= \rho_0\, \bar{q}(s)\frac{(s^2+f^2)^{\frac{1}{2}}(s^2+N^2)^{\frac{1}{2}}}{s}\delta(x)\, \delta(z'), \qquad (2.13)$$

using the property $\delta(ax) = \delta(x)/|a|$ (Carrier, Krook & Pearson 1966, p. 320). It should be noted that the solution to (2.13) is not unique up to a constant function of s (e.g. Davies 1978, p. 162). However, since the constant is a function of s only then $\bar{p}_x$ and

$\bar{p}_z$, and hence the velocity components, are unique. Ignoring then the undetermined function of s, the solution to (2.13) is well known (e.g. Davies 1978, p. 164) and is given by (in terms of the original variables x and z)

$$\bar{p} = \rho_0 \bar{q}(s) \frac{(s^2+f^2)^{\frac{1}{2}}(s^2+N^2)^{\frac{1}{2}}}{2\pi s} \log\left[\left(x^2+\frac{s^2+N^2}{s^2+f^2}z^2\right)^{\frac{1}{2}}\right]. \tag{2.14}$$

The velocity components (in Laplace space) can be found from (2.9) and (2.10) and are given by

$$\bar{u} = \frac{-1}{2\pi}\bar{q}(s)\left(\frac{s^2+N^2}{s^2+f^2}\right)^{\frac{1}{2}} \frac{x}{x^2+(s^2+N^2/s^2+f^2)\,z^2}, \tag{2.15}$$

$$\bar{w} = \frac{-1}{2\pi}\bar{q}(s)\left(\frac{s^2+N^2}{s^2+f^2}\right)^{\frac{1}{2}} \frac{z}{x^2+(s^2+N^2/s^2+f^2)\,z^2}. \tag{2.16}$$

Further using (2.6) the transformed azimuthal velocity is determined:

$$\bar{v} = \frac{f}{2\pi s}\bar{q}(s)\left(\frac{s^2+N^2}{s^2+f^2}\right)^{\frac{1}{2}} \frac{x}{x^2+(s^2+N^2/s^2+f^2)\,z^2}. \tag{2.17}$$

The density perturbation $\bar{\rho}$ can be calculated from (2.8) using (2.16). The above solution can be used to generate, via the method of images, various flow configurations in the same way that source/sink solutions are used in potential flow theory to, for example, find the velocity field for potential flow around a circular cylinder. An example of this is presented in McDonald (1990) which describes the circulation induced by the entrainment of a plane plume in a rotating stratified fluid, where the plume is represented by an appropriate distribution of sources and sinks.

It is a simple matter to check that the velocity components (2.15)–(2.16) satisfy the initial conditions. For small times $s \gg f, N$ in which case the Laplace inversion yields, for the horizontal velocity,

$$u(x,z,0^+) = -\frac{q(0)}{2\pi}\frac{x}{x^2+z^2}. \tag{2.18}$$

This is just the horizontal component of the velocity field for potential flow induced by a line sink in two dimensions. Similarly it can be shown that the vertical velocity w is also equivalent to that caused by a line sink in potential flow and further $v \rightarrow 0$ in this limit. In principle, for a given $q(t)$, the Laplace inverse of the above can be performed to give the time-dependent velocities. Before a discussion is given on inverting the above transforms for a particular $q(t)$, it is possible to deduce several properties of u, v and w in various special cases.

2.2. *Equal rotational and stratification effects*

When both rotational and buoyancy effects are equal in magnitude (i.e. $f = N$) the transformed velocities simplify considerably and inversion is a simple task. Putting $N = f$ into (2.15)–(2.17) and carrying out the inversion yields

$$u = -\frac{q(t)}{2\pi}\frac{x}{x^2+z^2}, \tag{2.19}$$

$$2 = -\frac{q(t)}{2\pi}\frac{z}{x^2+z^2}, \tag{2.20}$$

$$v = \frac{f}{2\pi}\left(\int_0^t q(u)\,\mathrm{d}u\right)\frac{x}{x^2+z^2}, \tag{2.21}$$

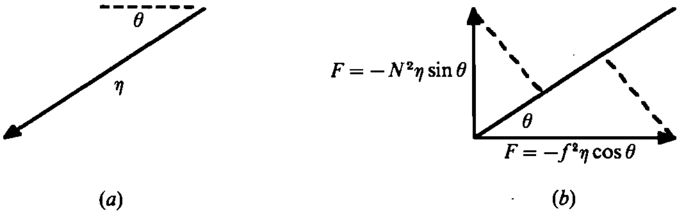

FIGURE 2. (*a*) The initial displacement of a fluid particle a small distance η at an angle θ below the horizontal as it is drawn towards the sink. (*b*) The resulting vertical force due to stratification and horizontal force due to rotation. The dashed lines indicate the components of the rotational and stratified induced forces in a direction perpendicular to the initial displacement. The resultant force perpendicular to the original path is $(N^2-f^2)\,\eta\cos\theta\sin\theta$.

where the integral in (2.21) is required to be bounded as $t\to\infty$, which implies that the amount of fluid removed by the sink is finite. This represents potential flow due a line sink of varying strength $q(t)$ in the (x,z)-plane with a superimposed azimuthal velocity parallel to the sink given by (2.21). This azimuthal velocity, or swirl, is a consequence of the conservation of angular momentum since as a fluid particle approaches the sink it must increase its swirl velocity in order to keep its angular momentum constant.

The above result can be explained by considering the forces acting on an individual fluid particle. A fluid particle in a stratified fluid of constant buoyancy frequency N when displaced a small vertical distance Δz experiences a restoring force per unit mass of $F_z=-N^2\Delta z$. By analogy, a particle in a rotating fluid displaced a small distance Δx horizontally experiences a restoring force per unit mass $F_x=-f^2\,\Delta x$ (see figure 2). Thus for a rotating stratified fluid, a particle displaced a small distance η at an angle θ to the horizontal experiences a transverse force to its path of magnitude $F=(N^2-f^2)\,\eta\cos\theta\sin\theta$. Hence if $f=N$ there is no transverse force acting on the particle and so there is no deviation from its initial path, which in this case is the potential flow pattern.

2.3. *Zero rotation*

If the fluid is stratified but not rotating then $f=0$ and so by (2.17) $v=0$ and the other velocities become (in Laplace space), after some rearrangement,

$$\bar{u}=\frac{-1}{2\pi}\bar{q}(s)\frac{(s^2+N^2)^{\frac{1}{2}}}{x^2+z^2}\frac{sx}{x^2+\omega^2}, \qquad (2.22)$$

$$\bar{w}=\frac{-1}{2\pi}\bar{q}(s)\frac{(s^2+N^2)^{\frac{1}{2}}}{x^2+z^2}\frac{sz}{s^2+\omega^2}, \qquad (2.23)$$

where $\omega^2=N^2z^2/(x^2+z^2)$. Choosing $q(t)=qH(t)$, which represents a sink of constant strength so that $\bar{q}(s)=q/s$, the Laplace transform portion of (2.22) can be written as

$$\frac{(s^2+N^2)^{\frac{1}{2}}}{s^2+\omega^2}=\frac{1}{(s^2+N^2)^{\frac{1}{2}}}+\frac{N^2-\omega^2}{(s^2+N^2)^{\frac{1}{2}}(s^2+\omega^2)}.$$

Applying the convolution theorem of the Laplace transforms the above can be inverted to give the following time-dependent horizontal velocity:

$$u(x,z,t)=\frac{-q}{2\pi}\frac{x}{x^2+z^2}\left(J_0(Nt)+\frac{N^2-\omega^2}{\omega}\int_0^t\sin\left[\omega(t-\tau)\right]J_0(N\tau)\,\mathrm{d}\tau\right), \qquad (2.24)$$

where J_0 is the zeroth-order Bessel function. This solution is identical to that obtained by Koh (1966*b*) for the two-dimensional sink flow of stratified non-rotating fluid. In the limit $t \rightarrow \infty$, (2.24) reduces to (ignoring decaying oscillations)

$$\begin{aligned} u(x,z,t) &= \frac{-q}{2\pi}\frac{x}{x^2+z^2}\frac{N^2-\omega^2}{\omega}\lim_{t\rightarrow\infty}\int_0^t \sin\left[\omega(t-\tau)\right] J_0(N\tau)\,\mathrm{d}\tau \\ &= \frac{-q}{2\pi}\frac{x}{x^2+z^2}(N^2-\omega^2)^{\frac{1}{2}}\lim_{t\rightarrow\infty}\frac{\sin\omega t}{\omega}. \end{aligned} \tag{2.25}$$

If the velocity field is interpreted as a distribution in the limit $t \rightarrow \infty$, then

$$\sin \omega t/\omega \rightarrow \pi\delta(\omega),$$

where δ is the Dirac delta function, and hence $u \rightarrow -\frac{1}{2}q\delta(z)$ (the factor of $\frac{1}{2}$ arises since there are two oppositely directed jets for $x < 0$ and $x > 0$). The vertical velocity vanishes in the same limit. Thus according to linearized theory, in the absence of rotation the flow of an inviscid, non-diffusive stratified fluid eventually collapses to a horizontal line jet at the level of the sink, a conclusion also reached by Koh (1966*b*).

3. Transient nature of the sink flow

Equations (2.15) and (2.16) suggest the existence of the following stream function:

$$\bar{\psi}(x,z,s) = -\frac{\bar{q}(s)}{2\pi}\arctan\left[\left(\frac{s^2+N^2}{s^2+f^2}\right)^{\frac{1}{2}}\frac{z}{x}\right],$$

where $\bar{u} = \bar{\psi}_z$ and $\bar{w} = -\bar{\psi}_x$. For large times such that $t \gg f^{-1}, N^{-1}$, then $s \ll f, N$ the inversion of the above transform becomes straightforward. For the case of constant sink strength, i.e. $q(t) = qH(t)$ the inversion yields the stream function

$$\psi(x,z) = \frac{q}{2\pi}\arctan\left[\frac{Nz}{fx}\right].$$

Thus the presence of both rotation and stratification imply that, after times large compared to both the inertial and buoyancy periods, a steady state in the (x,z)-plane is reached. This is in contrast to the case when there is no rotation where it was shown that the flow towards the sink keeps collapsing indefinitely. Further, at this steady state if the horizontal lengthscale is given by L then it follows from the above stream function that the vertical lengthscale, δ, is given by $\delta = fL/N$. Hence the vertical lengthscale grows linearly with horizontal distance from the sink. Superimposed on the steady-state stream function is, from (2.6), a swirl velocity v which increases linearly with time, i.e. $v \sim Ntq/L$. Similarly from (2.8) the density perturbation ρ increases linearly with time. Given this increase in v the linear equation (2.6) (and (2.8)) remains valid until $v_t \sim uv_x$ or until times such that $ft \sim F^{-1}$, which is large by assumption.

The approach to the steady state described above is examined in detail in the following section by carrying out the Laplace inversions in detail and the transient behaviour is interpreted in terms of inertio–gravity wave radiation by the sink.

3.1. *General solution*

The behaviour of a rotating fluid is examined for the case of a sink of constant strength q which is suddenly turned on at $t = 0$. Thus $q(t) = qH(t)$, where $H(t)$ is the unit step function, and hence $\bar{q}(s) = q/s$. In order to find the resultant velocity field, for this choice of sink behaviour, the task is to carry out the inverse Laplace transform on $\bar{u}$ where

$$\bar{u} = \frac{-q}{2\pi}\frac{x}{x^2+z^2}\frac{(s^2+f^2)^{\frac{1}{2}}(s^2+N^2)^{\frac{1}{2}}}{s(s^2+\omega^2)}, \tag{3.1}$$

and $\omega^2 = (f^2x^2+N^2z^2)/(x^2+z^2)$. Similar inversions are required to find the other velocity components. It will be assumed from here on that $f < N$, as is the case in most oceanographic and limnological situations. By symmetry, if $f > N$ then the following results still apply with f and N, and x and z interchanged. Note also that, from the definition of ω, it follows that $f \leqslant \omega \leqslant N$ for all x and z.

Writing

$$\frac{(s^2+f^2)^{\frac{1}{2}}(s^2+N^2)^{\frac{1}{2}}}{s(s^2+\omega^2)} = \left(\frac{s}{(s^2+f^2)^{\frac{1}{2}}}+\frac{f^2}{s(s^2+f^2)^{\frac{1}{2}}}\right)\frac{(s^2+N^2)^{\frac{1}{2}}}{s^2+\omega^2},$$

the inverse of (3.1) may be obtained using the result (2.24) and the convolution theorem for Laplace transforms. Applying standard results for inverting Laplace transforms the result is

$$u(x,z,t) = \frac{-q}{2\pi}\frac{x}{x^2+z^2}\left(\delta(t)-fJ_1(ft)+f\int_0^{ft}J_0(\alpha)\,\mathrm{d}\alpha\right)$$
$$*\left(J_0(Nt)+\frac{N^2-\omega^2}{\omega}\int_0^t \sin\,(\omega(t-\tau)]\,J_0(N\tau)\,\mathrm{d}\tau\right), \tag{3.2}$$

where $\delta(t)$ is the Dirac delta function, J_1 is the first-order Bessel function and $*$ denotes the convolution operator, which for two functions $f(t)$ and $g(t)$ is defined by

$$g*f = \int_0^t g(\tau)f(t-\tau)\,\mathrm{d}\tau.$$

For times small compared to the inertial period, $ft \ll 1$ the left-hand side of the convolution reduces to the delta function and (3.2) becomes identical to (2.24). This implies that initially the sink flow proceeds like that in the absence of rotation: that is, the initial potential flow proceeds to collapse to a horizontal withdrawal layer structure due to the action of buoyancy-induced forces. This collapse continues until time $t \sim f^{-1}$ when the other terms on the left-hand side of the convolution can no longer be ignored, i.e. when rotational effects become important. For even larger times, such that $ft \gg 1$ (and hence, by assumption, $Nt \gg 1$), it can be shown from (3.2) that a steady-state velocity field is obtained in the (x,z)-plane. This steady state will be discussed in more detail later in this paper.

Although mathematically precise, the temporal evolution and eventual steady state of the sink flow described by (3.2) is physically unclear. The next section examines the development of the flow in the context of inertio-gravity wave radiation by the sink. In particular the radiation of waves by an impulsive disturbance (i.e. sink strength represented by a Dirac delta function) is studied and these results are used to discuss the time development of sink flow with a sink of constant strength.

3.2. *Impulsive disturbance*

The response of a rotating stratified fluid is examined in detail when $q(t) = q\delta(t)$, which corresponds physically to switching on and off the sink if a very short space of time. Specifically, the velocity field is examined at large times after this event, i.e. times larger than f^{-1}. With this choice of sink flow and realizing that the Laplace transform of the delta function is unity, in order to find u it is required to evaluate the contour integral given by

$$I = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{e}^{st} \frac{(s^2+f^2)^{\frac{1}{2}}(s^2+N^2)^{\frac{1}{2}}}{s^2+\omega^2} \mathrm{d}s. \tag{3.3}$$

The integral can be evaluated by closing the contour at infinity and evaluating the contributions to the integral from the enclosed singularities. However, since the transform part of the integrand in (3.3) tends to a constant as $s \rightarrow \infty$ there is a contribution to the integrand by the contour at infinity. This is of the delta function type and is not important for large times and is therefore ignored. Alternatively, the problem can be avoided by letting $q(t) = q[H(t) - H(t-a)]/a$, evaluating the inverse transform, and then taking the limit $a \rightarrow 0$. The integrand has simple poles at $s = \pm \mathrm{i}\omega$ and branch points at $s = \pm \mathrm{i}f$ and $s = \pm \mathrm{i}N$.

It is shown in the Appendix that the parts of the contour integral involving the branch cuts at $\pm \mathrm{i}N$ and $\pm \mathrm{i}f$ represent decaying oscillations at the buoyancy and inertial frequencies respectively which behave like $t^{-\frac{3}{2}}$ for large t. Therefore at large times ($t \gg f^{-1}$) the main contribution to the integral comes from the simple poles at $s = \pm \mathrm{i}\omega$. To find this contribution the residues at $s = \omega \mathrm{e}^{\pm \mathrm{i}\pi/2}$ are calculated, where the poles have been written in exponential form to avoid the ambiguity that occurs when branch points exist. The residue calculation is straightforward and the following velocity components are obtained:

$$u(x,z,t) = \frac{-qx}{2\pi(x^2+z^2)} \frac{(N^2-\omega^2)^{\frac{1}{2}}(\omega^2-f^2)^{\frac{1}{2}}}{\omega} \cos(\omega t), \tag{3.4}$$

$$w(x,z,t) = \frac{-qz}{2\pi(x^2+z^2)} \frac{(N^2-\omega^2)^{\frac{1}{2}}(\omega^2-f^2)^{\frac{1}{2}}}{\omega} \cos(\omega t), \tag{3.5}$$

$$v(x,z,t) = \frac{fqx}{2\pi(x^2+z^2)} \left(\frac{Nf}{\omega^2} + \frac{(N^2-\omega^2)^{\frac{1}{2}}(\omega^2-f^2)^{\frac{1}{2}}}{\omega^2} \sin(\omega t) \right). \tag{3.6}$$

Thus the impulsive disturbance sets up an oscillation in the (x,z)-plane with an azimuthal velocity parallel to the sink. The oscillations are a result of inertio–gravity waves of local frequency ω being radiated by the sink. The frequency ω can be written in polar coordinates as $\omega^2 = f^2 \cos^2\theta + N^2 \sin^2\theta$, where θ is the angle from the x-axis to the point of observation. The wavefronts are in the radial direction θ = constant and the lines of constant phase, given by $\phi = \omega t$, are straight lines fanning radially out from the origin. On the constant-phase lines as time increases, ω and hence θ (for $f < N$) must decrease accordingly. It follows that the lines of constant phase rotate towards the horizontal axis in all four quadrants. Stevenson (1973) experimentally observed similar behaviour of constant-phase lines when a cylinder is made to undergo an impulsive displacement in a stratified non-rotating fluid.

Since for inertio–gravity waves group velocity is perpendicular to the phase velocity, it follows that energy is propagated radially from the sink. The wavenumber of the oscillations is given by

$$k = |\nabla \omega t| = \frac{1}{r}\frac{\partial \omega t}{\partial r} = \frac{\cos\theta \sin\theta (N^2-f^2)t}{\omega r}. \tag{3.7}$$

A point at which a given wavenumber is found moves radially out with velocity

$$\frac{\cos\theta \sin\theta(N^2-f^2)}{\omega k}. \tag{3.8}$$

This is the group velocity for a wave in a fluid rotating with angular velocity $\frac{1}{2}f$ and stratified with buoyancy frequency N (see e.g. Gill 1982).

From (3.8), by maximizing the group velocity with respect to θ, the maximum group velocity or energy radiated occurs along the line $z = (f/N)^{\frac{1}{2}}x$ and is carried by waves of frequency $\omega = (fN)^{\frac{1}{2}}$. This also corresponds, not unexpectedly, to the maximum amplitude of oscillation of the fluid particles. The ratio f/N is thus of critical importance in determining the direction at which the maximum amount of energy is radiated. As $f \rightarrow 0$ an increasing percentage of the energy radiated by the sink is concentrated along the horizontal axis and is carried by waves of decreasing frequency. Only when $f = 0$ (i.e. when the fluid is not rotating) is the (maximum) energy carried at vanishing frequencies along the horizontal axis. The properties of such waves are discussed in detail in the context of Taylor column formation in a rotating homogeneous fluid in Bretherton (1967).

3.3. *Maintained sink flow*

Consider now a maintained sink flow in which $q(t) = qH(t)$. A similar residue calculation is performed as in the previous section for the poles at $s = \pm i\omega$ as well as the additional pole at $s = 0$. As before the contribution of the decaying oscillations (now $\sim t^{-\frac{1}{2}}$) represented by the branch points at $s = \pm if, \pm iN$ is ignored. The velocity components are

$$u = \frac{-qx}{2\pi(x^2+z^2)}\left(\frac{fN}{\omega^2} + \frac{(N^2-\omega^2)^{\frac{1}{2}}(\omega^2-f^2)^{\frac{1}{2}}}{\omega^2}\sin(\omega t)\right), \tag{3.9}$$

$$w = \frac{-qz}{2\pi(x^2+z^2)}\left(\frac{fN}{\omega^2} + \frac{(N^2-\omega^2)^{\frac{1}{2}}(\omega^2-f^2)^{\frac{1}{2}}}{\omega^2}\sin(\omega t)\right), \tag{3.10}$$

$$v = \frac{fx}{2\pi(x^2+z^2)}\left(\frac{fNt}{\omega^2} - \frac{(N^2-\omega^2)^{\frac{1}{2}}(\omega^2-f^2)^{\frac{1}{2}}}{\omega^3}\cos(\omega t)\right). \tag{3.11}$$

Consider also a similar calculation for the case of a non-rotating fluid in which $f = 0$. In this case there is no pole at $s = 0$ and the residue calculation yields

$$u(x,z,t) = \frac{-qx}{2\pi(x^2+z^2)}(N^2-\omega^2)^{\frac{1}{2}}\frac{\sin\omega t}{\omega}. \tag{3.12}$$

There is a similar expression for the vertical velocity w, and $v = 0$. Consider the behaviour of (3.12) for large times. If, in the limit $t \rightarrow \infty$, $\sin\omega t/\omega$ is interpreted as a distribution, i.e. $\sin\omega t/\omega \rightarrow \pi\delta(\omega)$ then (3.12) reduces to (2.25) and thus the velocity field evolves into a line jet along the x-axis. Treating the velocity field (3.9)–(3.11) as distributions thus provides a convenient way of interpreting their behaviour at large times.

Consider the case when both rotation and stratification are present. Interpreting these velocity fields as distributions as $t \rightarrow \infty$ both $\cos\omega t$ and $\sin\omega t \rightarrow 0$ since now $\omega > 0$. Thus the u and w velocities eventually approach steady state because there are no waves of zero frequency. It is the presence of the low-frequency cutoff f in the rotating case which is the cause of the difference to the non-rotating case.

This result can be explained in terms of radiation of inertio-gravity waves. First the case of no rotation is considered and is then contrasted with the case when both rotation and stratification are present. In the absence of rotation the phase of the internal waves, from (3.7), is given by $\phi = Nt\sin\theta$ and the wavenumber $k = Nt\cos\theta/r$. Thus for a maintained source of internal waves, at any location in space not

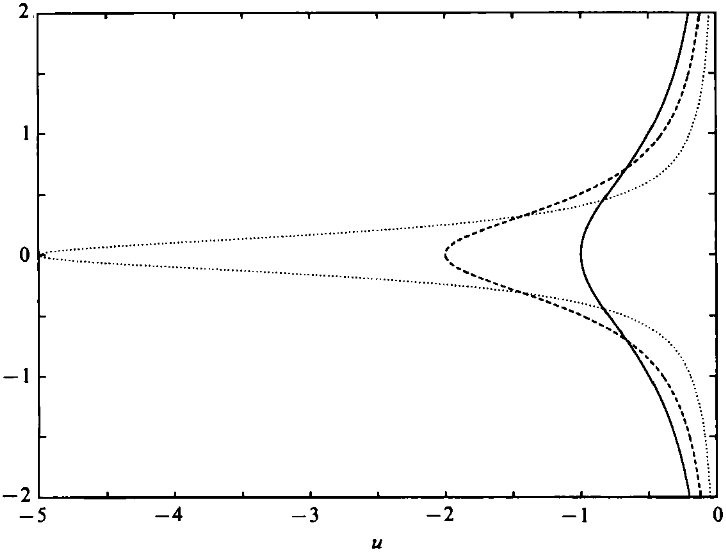

FIGURE 3. Horizontal velocity profiles at different values of the ratio f/N at a horizontal distance L from the sink: $f/N = 1.0$ (——), $f/N = 0.5$ (– – –), $f/N = 0.2$ (· · · · · ·). The velocity has been non-dimensionalized by $q/2\pi L$ and x and z by L.

on the x-axis (i.e. $\theta \neq 0$) waves emitted by the sink of all phases and wavenumbers will eventually be observed. These waves will destructively interfere (Bretherton 1967) and the velocity field approaches a steady value. At $\theta = 0$ only waves of the same phase ($\phi = 0$) will be observed arriving with successively smaller wavelength and constructive interference will occur. The velocity then increases with time along $\theta = 0$ eventually giving a delta function type velocity field. When both rotation and stratification are present, $\phi = \omega t$, and nowhere is there a location where the phase of each wave arriving is the same, since $\omega > 0$. Further from (3.7) the full range of wavenumbers will be observed at any given location. Hence destructive interference occurs and a steady state in the (x, z)-plane will eventually be observed. This steady state forms at times such that $\omega t \gg 1$ or $t \gg f^{-1}$ since the minimum inertio-gravity wave frequency is the inertial frequency.

The nature of the steady-state flow can be visualized by noting that in a coordinate frame in which the vertical axis is stretched by a factor of f/N the streamlines are identical to that caused by a sink in potential flow. Figure 3 shows the steady-state horizontal velocity profiles given by (3.9) at large times for various values of f/N. There is an increasing tendency for a well-defined narrow jet to form as f/N decreases, i.e. as stratification dominates over rotation. In the limit $f/N \to 0$ the horizontal velocity collapses to a line jet as expected.

It has been shown that the flow of an inviscid, non-diffusive, rotating and stratified fluid towards a sink collapses to a horizontal withdrawal layer structure if $f < N$. For $f > N$ the withdrawal layer is vertical. The eventual steady state is dependent on the ratio f/N. However, real fluids are viscous and the Froude number is finite, meaning that in the flow of a stratified non-rotating fluid towards a sink eventually either the effects of viscosity, diffusivity or nonlinearity, or a combination thereof, become important. The importance of rotation on the withdrawal problem for a real fluid may then be judged on the relative sizes of the withdrawal layer thicknesses produced by a balance between stratification and either viscous, nonlinear or rotational effects. This is done in the next section through the use of scaling analysis.

4. Viscous and nonlinear effects: scale analysis

In this section scales are obtained for withdrawal layer thicknesses and a classification scheme derived for the associated force balance regimes for the flow of a rotating stratified fluid into a line sink. Equations (2.1)–(2.3) form the starting point of the scaling analysis. As before the fluid is unbounded and is withdrawn through a link sink (in the y-direction). Hence there is no dependence of the flow variables on the transverse y-coordinate. It is further assumed that all flow variables, in particular v and ρ have reached steady state. The fluid is stratified with constant buoyancy frequency N. The Boussinesq approximation is used, i.e. the density term multiplying the inertial terms is approximated by the density at the level of the sink, i.e. $\rho_0(z) = \rho_0(0)$. The perturbation to the undisturbed density profile $\rho_0(z)$ induced by the motion is denoted by $\rho(x,z,t)$ and it is assumed that $|\rho| \ll \rho_0$. Conservation of mass (2.3) enables the introduction of a stream function ψ defined by $u = \psi_z, w = -\psi_x$. Eliminating the pressure from (2.1) the equation for the azimuthal vorticity, $\nabla^2\psi$, is

$$\underset{[a]}{\mathrm{J}(\psi, \nabla^2\psi)} - \underset{[b]}{fv_z} = \underset{[c]}{g/\rho_0 \rho_x} + \underset{[d]}{\nu\nabla^4\psi}, \tag{4.1}$$

where J is the Jacobian operator defined by $\mathrm{J}(a,b) = a_x b_z - b_x a_z$ and ν is the kinematic viscosity defined by $\nu = \mu/\rho_0(0)$. The letters used to identify various terms are for later convenience. The equations for the azimuthal momentum and perturbation density may also be written in terms of the stream function, yielding

$$\underset{[e]}{\mathrm{J}(\psi, v)} + \underset{[f]}{f\psi_z} = \underset{[g]}{\nu\nabla^2 v}, \tag{4.2}$$

and

$$\underset{[h]}{\mathrm{J}(\psi, \rho)} = \underset{[i]}{-\rho_0 N^2/g\, \psi_x} + \underset{[j]}{\kappa\nabla^2\rho}. \tag{4.3}$$

Only the case $f < N$ is considered here, as it represents most naturally occurring systems. This means that, as shown in the previous section, stratification initially dominates over rotation and the initial potential flow collapses to a horizontal withdrawal layer structure. As the flow collapses the vertical lengthscale decreases and so diffusion of momentum and species becomes more important. Further, the induced horizontal velocity in the withdrawal layer increases as the layer becomes narrower, meaning that inertial effects become more important. Thus the effects of nonlinearity and diffusion of vorticity and species are included and these will be important in determining the final steady-state structure. The balance which occurs at steady state depends on the relative magnitude of the nonlinear, Coriolis and viscous terms in the vorticity equation. In order to find the conditions under which various balances apply it is assumed that the baroclinic production of vorticity term [c] is dominant since this is the term responsible for the formation of the horizontal withdrawal layer. As the flow collapses to a horizontal jet structure the terms [a], [b] and [d] in the vorticity equation increase in magnitude until one of these terms balances with [c] at steady state. Further, it is assumed that at steady state the vertical lengthscale δ (the scale for the withdrawal layer thickness) is much smaller than the horizontal lengthscale L, i.e. $\delta \ll L$. This means that the Laplacian, ∇^2, and biharmonic, ∇^4, operators in (4.1)–(4.3) can be approximated by $\partial^2/\partial z^2$ and $\partial^4/\partial z^4$ respectively.

Imberger *et al.* (1976) derived a classification scheme for the above problem for the case when $f = 0$ based on shear wave dynamics. Such waves were of vanishing frequency but non-zero group velocity and definite modal structure because of being contained in a domain that is bounded in the vertical. When the withdrawal flow is initiated the initial potential flow is modified by shear waves propagating away from the sink leaving the withdrawal layer structure in their wake. A steady-state withdrawal layer is achieved when either the shear waves are unable to propagate against the withdrawal current or the shear waves are diffused by the action of viscosity. For a sink of strength q and a fluid with kinematic viscosity ν, in order to distinguish between the two cases they introduced the parameter

$$R = FGr^{\frac{1}{3}},$$

where F is the Froude number which measures the relative strengths of inertial and buoyancy effects and $Gr = N^2L^4/\nu^2$ is the Grashof number which measures the relative strengths of buoyancy and viscous effects. Thus R measures the relative strength of terms [a] and [d]. The following scales for the withdrawal layer thickness were found by Imberger *et al.* (1976):

$$\text{if} \quad R > 1 \quad \text{then} \quad \delta/L \sim F^{\frac{1}{2}}, \tag{4.4}$$

$$\text{or} \quad Pr^{-\frac{5}{6}} < R < 1 \quad \text{then} \quad \delta/L \sim Gr^{-\frac{1}{6}}R^{\frac{1}{5}}, \tag{4.5}$$

$$\text{or} \quad R < Pr^{-\frac{5}{6}} \quad \text{then} \quad \delta/L \sim Gr^{-\frac{1}{6}}Pr^{-\frac{1}{6}}, \tag{4.6}$$

where $Pr = \nu/\kappa$ is the Prandtl number which is assumed to be of order greater than, or equal to, one.

The parameter R is also used in this study. When rotation is included in the analysis a further parameter is required to delineate the extra possible force balances. It was shown in §3 that if the horizontal lengthscale is L for a balance between buoyancy and rotation then the vertical lengthscale is $\delta = fL/N$. That is, if the flow of a stratified rotating fluid towards the sink is in a thermal wind balance then the withdrawal layer thickness increases linearly from the sink. Imberger *et al.* (1976) (for $Pr = O(1)$) showed that, in the absence of rotation, the withdrawal layer thickness is constant from the sink if buoyancy balances inertia, or increases like $L^{\frac{1}{3}}$ if buoyancy balances viscosity. The growth of the withdrawal layer thicknesses in the various force balance regimes is shown schematically in figure 4. A convenient parameter to use to delineate various possible force balance regimes is one that measures the relative withdrawal layer thicknesses of the buoyancy–viscosity and the buoyancy–rotation balances. Such a parameter is

$$\gamma = \frac{fLN^{-1}}{LGr^{-\frac{1}{6}}} = \frac{f}{N}Gr^{\frac{1}{6}}.$$

Hence $\gamma > 1$ means that the rotation-induced layer thickness is larger than that of the viscous layer and rotation may have a significant effect on the withdrawal dynamics (depending on the magnitude of the inertial layer which may be larger still). For $\gamma < 1$ rotation will have no effect since the viscous withdrawal layer is always of greater thickness than the rotational layer.

The parameter γ may also be interpreted in terms of timescales by writing

$$\gamma = \frac{f}{N}Gr^{\frac{1}{6}} = \frac{N^{-1}Gr^{-\frac{1}{6}}}{f^{-1}}.$$

This expression for γ may be interpreted as follows: the initial collapse proceeds as if there were no rotation and so from §3 the time taken for a given wavenumber k

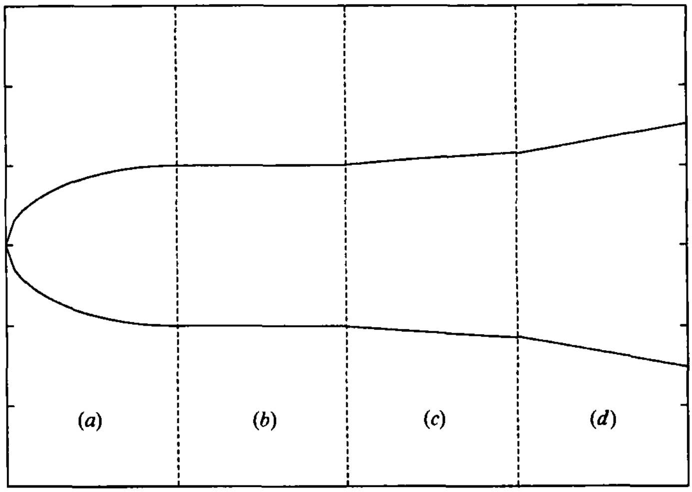

FIGURE 4. Idealized withdrawal layer, showing different rates of growth in each of the different force balance regimes: (*a*) potential flow, buoyancy unimportant; (*b*) inertial–buoyancy regime, δ constant; (*c*) viscous–buoyancy, $\delta \sim x^{\frac{1}{3}}$; (*d*) rotational–buoyancy, $\delta \sim x$.

$\gamma > Pr^{\frac{1}{3}}$	$R > \gamma^2$	$\delta \sim (q/N)^{\frac{1}{2}}$ (i)
	$\gamma^{-1} < R < \gamma^2$	$\delta \sim Lf/N$ (ii)
	$\gamma^{-1}Pr^{-\frac{1}{2}} < R < \gamma^{-1}$	$\delta \sim L(q/\nu)(f^2/N^2)$ (iii)
	$R < \gamma^{-1}Pr^{-\frac{1}{2}}$	$\delta \sim L(f/N)Pr^{-\frac{1}{2}}$ (iv)
$1 < \gamma < Pr^{\frac{1}{3}}$	$R > \gamma^2$	$\delta \sim (q/N)^{\frac{1}{2}}$
	$\gamma^{-1} < R < \gamma^2$	$\delta \sim Lf/N$
	$\gamma^{-\frac{5}{2}} < r < \gamma^{-1}$	$\delta \sim L(q/\nu)(f^2/N^2)$
	$Pr^{-\frac{5}{6}} < R < \gamma^{-\frac{5}{2}}$	$\delta \sim (\nu qL/N^2)^{\frac{1}{5}}$ (v)
	$R < Pr^{-\frac{5}{6}}$	$\delta \sim (\nu\kappa)^{\frac{1}{6}}(L/N)^{\frac{1}{3}}$ (vi)
$\gamma < 1$	$R > 1$	$\delta \sim (q/N)^{\frac{1}{2}}$
	$Pr^{-\frac{5}{6}} < R < 1$	$\delta \sim (\nu qL/N^2)^{\frac{1}{5}}$
	$R < Pr^{-\frac{5}{6}}$	$\delta \sim (\nu\kappa)^{\frac{1}{6}}(L/N)^{\frac{1}{3}}$

TABLE 1. The various force balance regimes

to reach a given location (r, θ) is given by $t = kr/N\cos\theta$. Consider a point of observation on the horizontal axis with $r = L$ and $\theta = 0$. Then $t \sim kL/N$. The time taken for a wave of wavenumber k to diffuse through the action of viscosity is given by $t \sim 1/k^2\nu$. Combining these gives an estimate for the viscous decay time of $t \sim N^{-1}Gr^{\frac{1}{3}}$ for an internal wave to decay at a distance L from the sink. Now $t \sim f^{-1}$ is the time for rotational effects to become important and thus γ is the ratio of these two timescales. Thus for $\gamma > 1$ the effect of rotation is felt before the internal waves radiated by the sink have had time to dissipate, and vice versa for $\gamma < 1$.

Scales for the withdrawal layer thickness, δ, arise from balancing [*c*] with [*a*], [*b*] or [*d*] and requiring that the remaining terms in the vorticity equation are small. Scales for the azimuthal velocity come from balancing [*f*] with either [*e*] or [*g*] and scales for the density perturbation come from balancing [*i*] with either [*j*] or [*h*]. The Prandtl number is assumed to be large. The scales may be split into three groups depending on the relative magnitude of γ and Pr and are shown in table 1.

The six possible different scales for the withdrawal layer thickness are indicated in table 1 by (i)–(vi). A convenient way to visualize the above scales is on a log–log (base

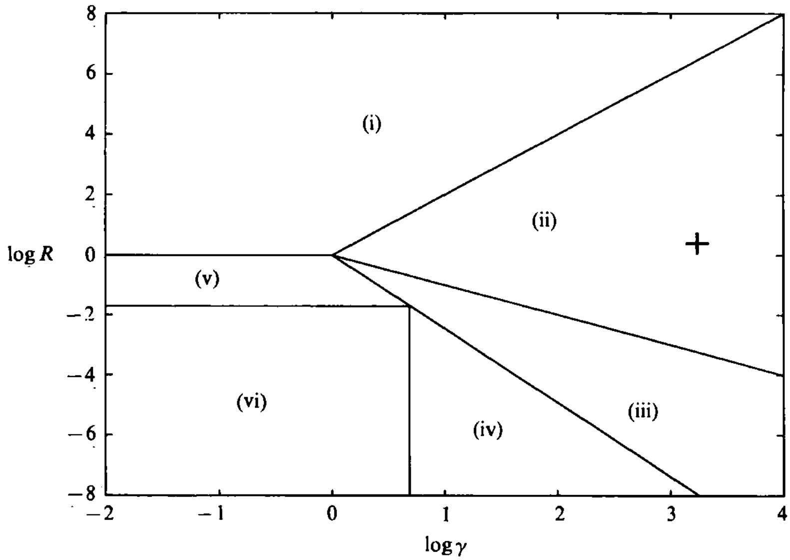

FIGURE 5. Log–log plot of γ versus R, for $Pr = 7.86$, identifying the various force balance regimes. The location of the Wellington Reservoir is indicated by a (+).

e) graph as in figure 5 (with $Pr = 7.86$) which shows a plot of $\log\gamma$ versus $\log R$. The various withdrawal layer thicknesses are indicated in the plot.

From the above table and figure 5 the system can be divided into six regimes, (i)–(vi), each yielding a different scale for the withdrawal layer thickness. Therefore the inclusion of rotation and the effect of large Prandtl number result in a further three layer thicknesses in addition to the three non-rotating scales found by Imberger *et al.* (1976). Note that when $f = 0$ then $\gamma < 1$ and the scales reduce to the non-rotating scales (iv)–(vi) as expected. Note also that in the regimes which involve rotation, the withdrawal layer thickness always grows linearly from the sink. This linear growth is faster than the non-rotating cases of constant inertial withdrawal layer thickness (iv) and the $L^{\frac{1}{3}}$ growth of the viscous withdrawal layer (vi). Thus rotation, within the assumptions of this theory, will thicken the withdrawal layer at sufficiently large distance from the sink. The scales (i), (v) and (vi) when rotation is not important in determining the withdrawal layer thickness are discussed elsewhere in the literature. In particular, the scale (i) (convective–buoyancy balance) is the inner solution and (vi) (viscous–buoyancy balance) is the outer solution of Imberger (1972) who solved the problem of a stratified fluid flowing towards a sink in horizontal duct. Koh (1966*a*) also discusses the viscous–buoyancy balance (vi) for a fluid of infinite extent. The scale (v) is applicable for fluids of large Prandtl number in which convection of the stratifying species (but not momentum) is important in determining the withdrawal layer thickness and is discussed in Imberger *et al.* (1976). However, in the regimes (i), (v) and (vi), due to rotation there is now a superimposed azimuthal velocity of magnitude $O(fL)$, $O(fLR^{\frac{6}{5}})$ and $O(fLRPr^{-\frac{1}{6}})$ respectively.

A discussion of the new scales, where rotation is important, is given below.

(ii) $\delta \sim (f/N)L$. The withdrawal layer thickness increases linearly from the sink and is proportional to the ratio of the inertial to buoyancy frequencies. The balance in the vorticity equation is $[c] \sim [b]$. In other words baroclinic production of vorticity balances the production of vorticity due to the Coriolis force, i.e. the thermal wind balance. The scale for the azimuthal velocity comes from $[e] \sim [f]$. That is the production of azimuthal velocity which is $O(fL)$ as fluid moves towards the sink is balanced by advection of azimuthal velocity by the mean flow in the (x, z)-plane. Similarly, in equation (4.3) $[i] \sim [h]$, which states that the displacement of the isopycnals is due to advection by the vertical velocity. The density perturbation is $\rho \sim \rho_0(N^2L/g)(f/N)$.

(iii) $\delta \sim (qf^2/\nu N^2)L$. The withdrawal layer thickness increases linearly from the sink. The vorticity equation is in thermal wind balance. The scale for the azimuthal velocity comes from $[g] \sim [f]$. That is, the production of azimuthal velocity which is $O(R^2\gamma^2 fL)$ as fluid moves towards the sink is balanced by diffusion of azimuthal velocity by viscosity. In the density perturbation equation the balance is $[i] \sim [h]$ and $\rho \sim \rho_0(N^2L/g)(f/N)R\gamma$.

(iv) $\delta \sim (f/N)LPr^{-\frac{1}{2}}$. The withdrawal layer again increases its thickness linearly from the sink. The vorticity equation is in thermal wind balance. The production of azimuthal velocity by the mean horizontal flow is balanced by viscous diffusion, i.e. $[g] \sim [f]$ and $v \sim (R\gamma Pr^{-\frac{1}{2}} fL)$. The density equation is dominated by the diffusion of the stratifying species, i.e. $[i] \sim [j]$ and $\rho \sim \rho_0(N^2L/g)(f/N)R\gamma$.

The results of this section are now discussed with reference to the reservoir withdrawal problem. As an example illustrating the above scales consider the field data reported in Ivey & Imberger (1978) for the Wellington reservoir. They found that the measured value of the withdrawal layer thickness was greater than that predicted using non-rotating link sink theory. They explained this discrepancy by the fact that they used laminar values of viscosity rather than the appropriate turbulent viscosity. Another explanation is that point sink theory (Ivey & Blake 1985) is more appropriate given the fact that reservoir offtake structures more closely resemble point sink. This paper raises the third possibility that withdrawal layer thickening may be due to rotational effects. The horizontal lengthscale is taken to be $L = 5$ km from the offtake. The relevant data are $N = 0.022\ \mathrm{s}^{-1}$, $\nu = 1.1 \times 10^{-6}\ \mathrm{m}^2\ \mathrm{s}^{-1}$, $Pr = 7.86$, $q = 0.013\ \mathrm{ms}^{-1}$ and $f/N = 3.3 \times 10^{-3}$. This yields $R = 1.5$ and $\gamma = 26$. Figure 5, which shows the location in parameter space of the Wellington reservoir, indicates that $\delta \sim (f/N)L$, giving $\delta \sim 16$ m at a distance of 5 km from the offtake. Even though this relation is only an order of magnitude estimate (i.e. there is an $O(1)$ constant multiplying this estimate) such an estimate of 16 m is clearly too large when compared to the measured value of the withdrawal layer thickness of about 3 m (Ivey & Imberger 1978). This implies that line sink theory is inappropriate for the Wellington reservoir when it is generalized to include rotational effects. The problem arises from the fact the reservoir is a bounded basin. In particular, if one treats a reservoir as a narrow channel, the effect of sidewalls perpendicular to the y-axis are of significance in the reservoir problem since, as shown by Gill (1976), they may be sufficiently close together that rotational effects are limited by the zero-normal-velocity (i.e. $v = 0$) boundary condition on the walls. Indeed in the limit that the channel width goes to zero Gill showed that rotational effects are negligible. The withdrawal of a stratified fluid from a rotating channel of finite width is currently being investigated by the authors.

5. Conclusions

The solution to the initial-value problem for a line sink in an unbounded, inviscid, non-diffusive, rotating and stratified fluid is obtained in the zero-Froude-number limit. The solution shows that for $f < N$ the flow collapses to a horizontal withdrawal layer structure, the eventual steady flow stream function of which depends on the ratio f/N. In addition there is a superimposed azimuthal velocity and density perturbation which increases linearly with time. The approach to steady state can be interpreted in terms of the superposition of inertio–gravity waves radiated by the sink which arrive at a particular location with ever decreasing wavelength. The existence of a steady state can be attributed to the presence of both stratification and rotation which precludes energy being radiated by the sink at vanishing frequencies.

For a real fluid (i.e. a fluid governed by the full Navier–Stokes equations) in which $f < N$ the flow collapses to a horizontal withdrawal layer structure where the collapse is eventually halted by either rotational, nonlinear or viscous and diffusive effects. These effects are incorporated using scaling analysis. For $\gamma = (f/N)\,Gr^{\frac{1}{3}} < 1$ rotation has no effect on the withdrawal layer thickness. For $\gamma > 1$ the withdrawal layer thickness depends on the relative magnitude of R, γ and Pr.

For the case when buoyancy balances rotation the steady-state aspect ratio of f/N of the withdrawal layer obtained here was also obtained by Gill (1981) for the aspect ratio of intrusions in rotating stratified fluid. Gill started with the steady equations of motion and took advantage of the fact, as seen here, that the steady state is closely related to potential flow to generate his solutions. This aspect ratio has been observed recently by Rosenblum & Marmorino (1990) as that determining the thickness of layers of turbulence in the ocean.

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Appendix. Transient behaviour represented by the branch points

Here the contribution of the branch points at $s = \pm \mathrm{i}f$ and $s = \pm \mathrm{i}N$ to the integral I is examined where I is given by

$$I = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} \frac{(s^2+f^2)^{\frac{1}{2}}(s^2+N^2)^{\frac{1}{2}}}{s^2+\omega^2} \mathrm{d}s. \tag{A 1}$$

First consider the branch points at $s = \pm \mathrm{i}f$. The complex plane can be cut twice from $s = -\infty$ to $s = 0$ and the Laplace inversion contour can be wrapped around these cuts as shown in figure 6(*a*). Consider first the branch cut located at $s = \mathrm{i}f$ which is surrounded by the contour $\mathscr{C}_1$. The origin is shifted by introducing the variable $\xi = s - \mathrm{i}f$ and so the contribution of this branch cut is

$$I_1 = \frac{1}{2\pi i} \int_{\mathscr{C}_1} \mathrm{e}^{(\xi+\mathrm{i}f)t} \frac{(\xi^2+2\mathrm{i}f\xi)^{\frac{1}{2}}(\xi^2+2\mathrm{i}f\xi-f^2+N^2)^{\frac{1}{2}}}{\xi^2+2\mathrm{i}f\xi-f^2+\omega^2} \mathrm{d}\xi. \tag{A 2}$$

To integrate around the branch cut the integral is split into three parts, namely (see figure 6*b*) (*a*) counterclockwise around the circle of radius r from $-\pi$ to π; (*b*) below the cut the substitution $\xi = x\,\mathrm{e}^{-\mathrm{i}\pi}$ is made where $x: \infty \rightarrow r$; (*c*) above the cut introduce $\xi = x\,\mathrm{e}^{\mathrm{i}\pi}$ where $x: r \rightarrow \infty$. The contour shown in figure 6(*b*) shrinks onto the branch cut in the limit $r \rightarrow 0$. It is straightforward to show that the contribution to the integral I_1 around the circle radius r centred at $s = \mathrm{i}f$ vanishes in the limit as its radius tends to zero. Making these substitutions and after some algebra I_1 becomes

$$I_1 = \frac{\mathrm{e}^{\mathrm{i}ft}}{\pi} \int_0^{\infty} \mathrm{e}^{-xt} \frac{(-x^2+2\mathrm{i}fx)^{\frac{1}{2}}(x^2-2\mathrm{i}fx-f^2+N^2)^{\frac{1}{2}}}{x^2-2\mathrm{i}fx-f^2+\omega^2} \mathrm{d}x. \tag{A 3}$$

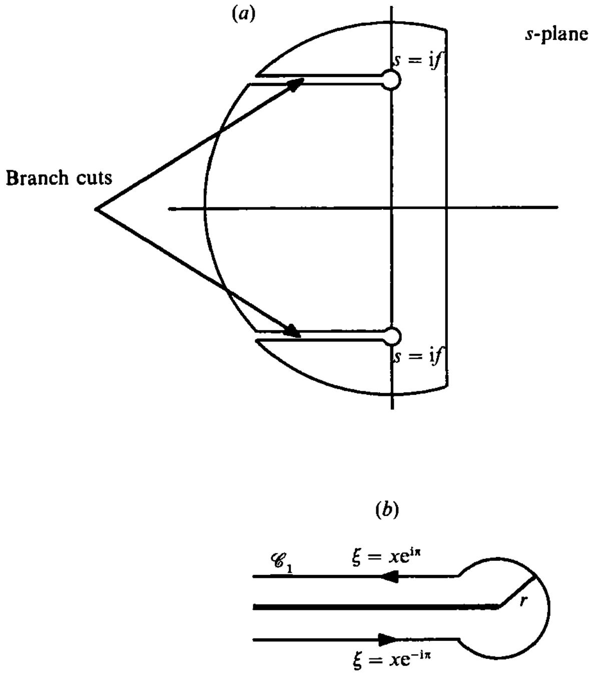

FIGURE 6. (*a*) Contour integral for the branch points at $s = \pm if$. (*b*) Substitutions around cut at $s = if$.

For large t the major part of the integral is contributed by values of x near the origin, owing to the rapid decrease in e^{-xt} with increase in x. Hence the leading-order term in I_1 as $t \to \infty$ can be found by finding the first-order term in the integrand as $x \to 0$. In this limit then

$$I_1 \sim e^{ift} i^{\frac{1}{2}} \int_0^\infty e^{-xt} x^{\frac{1}{2}} dx, \tag{A 4}$$

$$\sim \frac{e^{i(ft+\pi/4)}}{t^{\frac{3}{2}}}. \tag{A 5}$$

The contribution from the branch point at $s = -if$ can be found by changing i to $-i$ in (A 5) which gives

$$I_2 \sim \frac{e^{-i(ft+\pi/4)}}{t^{\frac{3}{2}}}. \tag{A 6}$$

The total contribution of the branch points $s = \pm if$ is then the sum of I_1 and I_2 and has behaviour $\cos(ft + \frac{1}{4}\pi)/t^{\frac{3}{2}}$ for large t. This represents a decaying, non-propagating oscillation at the inertial frequency. Similarly, it can be shown that the branch at $s = \pm iN$ represent a decaying oscillation at the buoyancy frequency with behaviour $\sin(ft + \frac{1}{4}\pi)/t^{\frac{3}{2}}$.

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